Algorithms for Solving the Catch Equation Forward and Backward in Time[1]

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Approximate solutions to the catch equation for the fishing mortality rate both forward and backward in time are obtained with an application of the diagonal Padé approximation of degree four to the exponential function. In either case the resulting approximation as well as Pope's estimate are shown to serve quite well as starting values for Newton's Method which is used to obtain a numerical solution of the catch equation. Convergence criteria for Newton's Method are discussed in each setting.

Key words: catch equation, Newton's method, Padé approximation, Pope's estimate

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Une application de l'approximation diagonale de Padé de degré quatre à la fonction exponentielle apporte des solutions approchées à l'équation de capture, donnant le taux de mortalité par pêche tant futur que passé. Dans un cas comme dans l'autre, nous démontrons que l'approximation qui en résulte, ainsi que l'estimation de Pope, peuvent assez bien servir de valeurs de départ dans la méthode de Newton donnant une solution numérique de l'équation de capture. Nous analysons dans chaque situation les critères de convergence de la méthode de Newton.

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IN the assessment of exploited marine fisheries a pair of coupled equations are used to calculate the fishing mortality rate and stock size in each time period for a given cohort. The first of these equations, the catch equation, can be written in the form

$$(1) \quad C_i/N_i = F_i(1 - \exp(-Z_i))/Z_i$$

or in the form

$$(2) \quad C_i/N_{i+1} = F_i(\exp(Z_i) - 1)/Z_i,$$

where, in the ith time period,

C_i = the catch,

N_i = the stock size at the beginning of the period,

F_i = the fishing mortality rate,

M_i = the natural mortality rate, and

$Z_i = F_i + M_i$ = the total mortality rate.

The other equation, the survival equation, relates the stock sizes N_i and N_{i+1} by

$$(3) \quad N_{i+1} = N_i \exp(-Z_i)$$

Several algorithms have been presented either in the literature or as computer programs to solve equations (1) and (2). Some of these make use of Newton's Method as do the algorithms presented herein. The program COHORT[3] solves the unnecessarily complicated Murphy form (Murphy 1965) of the catch equation with initial values arbitrarily chosen as the fishing mortality rate solved for in the adjacent time period.

Mesnil (1978) has also used Newton's Method on a simplified version of equation (2), but problems of convergence to an extraneous solution can arise with his technique where again initial values are not carefully selected.

Other algorithms make use of bracketing techniques. Miller (1977) and Doubleday (1975) have used false position methods to solve equations (1) and (2), respectively, after taking the square root of each side of these respective equations. The program MURPHY by P. K. Tomlinson (Abramson 1971) solves the Murphy form of the catch equation using the highly inefficient bisection method.

Newton's Method, in general, converges more rapidly (quadratically), if it does indeed converge, than do these bracketing methods, and is thus the better of these techniques if convergence to the desired solution is assured.

The objective of this work was to select carefully starting values for Newton's Method.

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Newton's Method

Newton's Method is an iterative technique to obtain a numerical solution of a functional equation $f(X) = 0$. The iterative scheme is of the form

$$(4) \quad X_{n+1} = X_n - f(X_n)/f'(X_n), \; n = 0, 1, 2, \ldots \; .$$

An initial estimate X_0 must be supplied to begin the iteration.

There exists a set of conditions attributed to Fourier which are sufficient to ensure that the sequence generated in (4) will converge to the unique zero of f in a given interval. These conditions are recorded below in keeping with their presentation in Burden et al. (1978).

Assume that f and f' are continuous on the closed interval $[a,b]$, and that $f''(x)$ exists for each x in the open interval (a,b), and that the following Fourier conditions are satisfied:

(i) $f(a) \cdot f(b) < 0$;
(ii) $f'(x)$ is strictly positive or strictly negative for all x in $[a,b]$;
(iii) $f''(x)$ does not change sign for all x in (a,b);
(iv) if c is the endpoint of $[a,b]$ at which $|f'(x)|$ is smallest then
$|f(c)/f'(c)| \leq b - a$.

Then the sequence generated by Newton's Method converges to the only zero of f in (a,b), provided that x_0 is chosen in (a,b).

Additionally, there are sufficient conditions that insure convergence provided that the initial estimate is chosen "close enough" to the desired zero of f. More precisely, if f is twice continuously differentiable on an interval $[a,b]$, if p in $[a,b]$ is such that $f(p) = 0$, and $f'(p) \neq 0$, there exists $\delta > 0$ such that Newton's Method generates a sequence (x_n) converging to p for any initial estimate x_0 in $[p - \delta, p + \delta]$.

Solution of the Catch Equation Forward in Time

There are at least two ways that a close estimate to the solution of (1) can be obtained. One such estimate can be had from Pope's (1972) approximation which can be written in terms of the total mortality as

$$(5) \quad Z_i = M_i - \log(1 - K\exp(M_i/2))$$

where $K = C_i/N_i$ and log represents the natural logarithm. However, the inequality

$$(6) \quad K < \exp(-M/2)$$

must hold if the argument of the logarithm is to be positive (the subscript i will be suppressed except where it is needed for clarity). For example, for M taking on all positive values less than or equal to unity equation (6) implies that

$$(7) \quad K < 0.61.$$

But, from equation (1), it is clear that

$$K \leq 1 - \exp(-Z),$$

so that, if

$$1 - \exp(-Z) < 0.61$$

then inequality (7) will hold. This last inequality is satisfied provided that $Z < 0.93$.

Another method of approximating the solution of equation (1) comes from an application of the diagonal Padé approximation of degree four to $\exp(-Z)$. This rational approximation is given by

$$(8) \quad R_4(Z) = (12 - 6Z + Z^2)/(12 + 6Z + Z^2).$$

If $\exp(-Z)$ is replaced by $R_4(Z)$ in equation (1), then, after some simplification, the quadratic equation

$$(9) \quad Z^2 - 6(2H - 1)Z + 12(HM + 1) = 0$$

results where, for simplicity, $H = 1/K$. Solutions of this equation are given by

$$(10) \quad Z = 3(2H - 1) \pm \sqrt{9(2H - 1)^2 - 12(HM + 1)}.$$

The solution obtained using the negative sign will be used for reasons discussed below. For the solutions in (10) to be valid the radicand must be nonnegative, that is,

$$9(2H - 1)^2 - 12(HM + 1) \geq 0.$$

This inequality is true if

$$(11) \quad H \geq (3 + M + \sqrt{(3 + M)^2 + 3})/6,$$

since the inequality

$$H \leq (3 + M - \sqrt{(3 + M)^2 + 3})/6$$

can be ruled out because $H \geq 0$.

For $M > 0$, the function

$$(12) \quad G(M) = (3 + M + \sqrt{(3 + M)^2 + 3})/6$$

is an increasing function of M, and thus if H is larger than the maximum of $G(M)$ on a given interval then inequality (11) holds on that interval. For instance, if as before, $0 \leq M \leq 1, G(M) \leq 1.39$, and hence $H \geq 1.39$, or

$$(13) \quad K \leq 0.72$$

is sufficient for (11) to be true. But (13) holds if $Z \leq 1.27$ so that the estimate stemming from equation (10) is known to be valid for a wider range of total mortality values than that given by equation (5).

To balance this favorable view of the estimate given by equation (10) it should be noted that, as the range of M becomes smaller, equation (5) is valid for a slightly larger range of total mortality values. For example, for $0 \leq M \leq 0.2$, the estimate given by (5) is viable for $Z \leq 2.35$ whereas (10) is known to hold for $Z \leq 2.09$. However, the initial estimate provided by (10) is, in general, more accurate than that given by (5) over a wide range of total mortalities as is indicated in Table 1. Hence, in the algorithms presented below, the initial estimate is taken from equation (10) where valid.

It is easy to see that if the plus sign is used in equation (10) and if $H \geq G(M)$ then

$$Z \geq 6H - 3 > 6G(M) - 3 \geq 3.46 \text{ for } M \geq 0.$$

This conservative lower bound precludes the use of this form for Z in a more or less realistic range. Furthermore, empirical evidence indicates that for large values of Z for which (11) holds (e.g. small F and large M) the above yields a gross overestimate. Thus, in (10) the negative sign is chosen and to reduce round-off error this initial estimate is written in the form

$$(14) \quad Z^{(0)} = A/(B + \sqrt{B^2 - A})$$

where

$$A = 12(HM + 1) \text{ and } B = 3(2H - 1).$$

In the event that (11) is not satisfied, Z will be relatively large. Indeed, if $H < G(M)$ then $Z > 1.2$ for any M. Hence $\exp(-Z)$ will be relatively small, and from (1) it is apparent that, in this case,

$$F/Z \approx C/N = K.$$

Solving for Z provides the following initial estimate

$$(15) \quad Z^{(0)} = M/(1 - K) = MH/(H - 1).$$

In the algorithm presented below H is compared to $G(M)$. If (11) holds then (14) is used to obtain the starting value; if not then (15) yields the initial estimate.

From equation (1) the function f can be defined by

$$(16) \quad f(Z) = (Z - M)((1 - \exp(-Z))/Z) - K.$$

The first derivative is given by

$$(17) \quad f'(Z) = [Z(Z - M)\exp(-Z) + M(1 - \exp(-Z))]/Z^2$$

A somewhat complicated expression for the second derivative indicates that f is twice continuously differentiable for all $Z > 0$. Furthermore, since $\exp(-Z) < 1$ for $Z > 0$ it is clear that $f'(Z) > 0$ for all $Z > 0$. Now $f(M) = -K < 0$, and $\lim_{Z\to\infty} f(Z) = 1 - K > 0$, together with the fact that f is increasing for $Z \geq M$ implies that f has a unique zero, say P, in an interval of the form $[M, D]$ for D sufficiently large. Furthermore, $f'(P) \neq 0$ and hence Newton's Method will converge to P provided that the initial estimate is selected sufficiently close to P. Although it is not possible to determine how close the starting value must be to P, experience indicates that the estimates given by (14) and (15) are accurate enough to insure convergence.

Newton's Method along with either (14) or (15) yields an algorithm for solving equation (1) which can now be outlined.

TABLE 1. Selected results of a comparison of the estimates for the fishing mortality rate obtained using Pope's estimate (eq. (5) and (18) and the Padé approximation (eq. (14) and (20)). The range for F is from 0.1 to 2.0 in steps of 0.1 with M ranging from 0.1 to 1.0 in the same increments. The absolute value of the relative error is given to four places for sake of comparison.

Selected values		Absolute relative error			
		Forward solution		Backward solution	
Natural mortality	Fishing mortality	Eq. (5)	Eq. (14)	Eq. (18)	Eq. (20)
0.1	0.1	0.0007	0.0000	0.0001	0.0000
	0.5	0.0049	0.0001	0.0030	0.0003
	1.0	0.0135	0.0019	0.0049	0.0021
	1.5	0.0278	0.0094	0.0060	0.0107
0.4	0.1	0.0021	0.0000	0.0019	0.0000
	0.5	0.0170	0.0004	0.0102	0.0006
	1.0	0.0506	0.0032	0.0178	0.0050
	1.5	0.1132	0.0131	0.0222	0.0218
0.7	0.1	0.0035	0.0001	0.0031	0.0001
	0.5	0.0275	0.0009	0.0163	0.0017
	1.0	0.0862	0.0049	0.0288	0.0107
	1.5	0.2130	0.0170	0.0364	0.0431
1.0	0.1	0.0049	0.0001	0.0044	0.0003
	0.5	0.0379	0.0014	0.0220	0.0040
	1.0	0.1238	0.0069	0.0388	0.0213
	1.5	0.3545	0.0211	0.0493	0.0884

ALGORITHM I. Suppose C_i and M_i are known and that N_i has been calculated in the previous step.

1. If $C_i = 0$ set $F_i = 0$ and go to step 11;
2. If $C_i \neq 0$ set $H = N_i/C_i$;
3. Calculate $G(M_i)$ using equation (12);
4. If $H \geq G(M_i)$ go to step 7;
5. Calculate $Z_i^{(0)}$ using equation (15);
6. Go to step 8;
7. Calculate $Z_i^{(0)}$ using equation (14);
8. For $n = 0, 1, 2$---, do:
   (i) Calculate $f(Z_i^{(n)})$ via equation (16)
   (ii) Calculate $f'(Z_i^{(n)})$ from expression (17)
   (iii) Set $Z_i^{(n+1)} = Z_i^{(n)} - f(Z_i^{(n)})/f'(Z_i^{(n)})$
   (iv) If $|Z_i^{(n+1)} - Z_i^{(n)}| \geq 5 \times 10^{-6}/Z_i^{(n+1)}$ go to (i);
9. Accept $Z_i = Z_i^{(n+1)}$ as an approximate solution of equation (1);
10. Set $F_i = Z_i - M_i$;
11. Set $N_{i+1} = N_i \exp(-(F_i + M_i))$.

Solution of the Catch Equation Backward in Time

As in the previous section there are at least two choices for a value of the total mortality to initiate Newton's Method. The first of these is again produced by Pope's (1972) estimate in the form

$$(18) \quad Z_i = \log(1 + K\exp(-M_i/2) + M_i$$

where $K = C_i/N_{i+1}$. The second approximation for Z_i is obtained by substituting the diagonal Padé approximation of degree four for $\exp(Z)$, which is the reciprocal of equation (8), into equation (2). Simplification yields, with

$$H = 1/K,$$

(19) $Z^2 - 6(2H + 1)Z + 12(HM + 1) = 0.$

For reasons similar to those mentioned above, the pertinent solution of (19) is given by

$$Z = 3(2H + 1) - \sqrt{9(2H + 1)^2 - 12(HM + 1)}.$$

or, to reduce round-off error

(20) $Z = A/(B + \sqrt{B^2 - A})$, with $B = 3(2H + 1)$ and $A = 12(HM + 1)$.

As before, this approximation is valid if $H \geq g(M)$ where now $g(M)$ is given by

$$g(M) = (-(M - 3) + \sqrt{(M - 3)^2 + 3})/6$$

which is a positive, increasing function of M for all $M \geq 0$. To obtain an idea of the range of total mortality for which (20) holds, note that

$$H \geq N_{i-1}/N_i = \exp(-Z_i).$$

Thus if,

$$\exp(-Z_i) \geq g(M),$$

That is, if

$$Z_i \leq -\log(g(M))$$

then

(21) $H \geq g(M)$.

For example, if $0 \leq M \leq 0.8$, then $g(M) \leq 0.1$ Thus, if $Z_i \leq \log(10) = 2.3$, then $H \geq \exp(-Z_i) \geq 0.1$ and (21) holds.

Thus the approximation given by (20) is valid for a wide range of values of total mortality. Inaccuracies occur, however, as Z increases because the diagonal Padé approximation of degree four estimates $\exp(Z)$ accurately when Z is small, but inaccurately as Z becomes larger. For example, if $Z \leq 0.5$, the error in this approximation is less than 7×10^{-5} while for $Z \leq 1.5$ the error is less than 0.053. It should be noted that this range of values compares favorably with that given by Pope (1972) where $M \leq 0.3$ and $F_i \leq 1.2$ is given as the range of values for which the approximation for F_i stemming from (18) yields adequate accuracy. However, equation (20) gives a better approximation than does (18) for smaller values of Z as is evident from Table 1. But (20) is not valid for larger values of Z, whereas (18) holds for all $Z \geq 0$; though accuracy is lost as Z increases. In the algorithm given here, if $H < 0.5$, that is, if $Z > 0.69$, the approximation in (18) is used instead of that given by (20). This choice is explained below.

The estimate for Z_i given in (20) provides a starting value for smaller values of total mortality while the approximation for Z_i given in (18) gives a starting value for larger values of Z_i to begin the iterative process for solving equation (2) numerically via Newton's Method. This yields the following algorithm to solve $f(Z) = 0$ where

(22) $f(Z) = (Z - M)(\exp(Z) - 1)/Z - K.$

ALGORITHM II. Suppose that C_i, and M_i are known and that N_{i+1} has been calculated in the previous step.

1. If $C_i = 0$ set $F_i = 0$ and proceed to step 9;
2. Set $H = N_{i+1}/C_i$ and $K = 1/H$;
3. If $H < 0.5$ go to step 6;
4. Use equation (20) to estimate Z_i and denote this value by $Z_i^{(0)}$;
5. Go to step 7;
6. Use equation (18) to approximate Z_i and denote this value by $Z_i^{(0)}$;
7. Use Newton's Method as outlined in Algorithm I to obtain an approximate solution, Z_i, for equation (2);
8. Set $F_i = Z_i - M_i$;
9. Set $N_i = N_{i+1}\exp(F_i + M_i)$.

It can now be shown that the function defined by (22) satisfies the Fourier conditions on the interval $[M, M + K]$. In order to proceed, the first and second derivatives of f are needed. These are given, respectively, by

(23) $f'(Z) = (Z(Z - M)\exp(Z) + M(\exp(Z) - 1))/Z^2,$

and

(24) $$f''(Z) = (Z^2(Z - M)\exp(Z) + 2M((Z - 1)\exp(Z) + 1))/Z^3$$

Obviously f and f' are continuous on $[M, M + K]$ and $f''(Z)$ exists for all Z in $(M, M + K)$. The Fourier conditions will be verified in the order presented above.

(i) Substitution gives

$$f(M) = -K < 0$$

and

$$f(M + K) = K(\exp(M + K) - 1)/(M + K) - K$$

Using the well known result

(25) $\exp(Z) > Z + 1$ for $Z > 0$,

it is easy to see that, in the present case,

$$(\exp(M + K) - 1)/(M + K) > 1,$$

so that

$$f(M + K) > 0.$$

(ii) Since $Z > M$ it is clear from an examination of (23) that $f'(Z) > 0$.

(iii) To show that $f''(Z) > 0$ for all Z in $(M, M + K)$ all that needs to be demonstrated is that the expression $(Z - 1)\exp(Z) + 1$ in (24) is positive. But this is an increasing function with a minimum of zero for $Z \geq 0$.

(iv) Since $f''(Z) > 0$, f' assumes its smallest value in magnitude at $Z = M$. Thus, to verify the last Fourier condition consider the expression

$$|f(M)/f'(M)| = KM/(\exp(M) - 1).$$

Again from (25), $M/(\exp(M) - 1) < 1$ so that

$$|f(M)/f'(M)| < K = M + K - M$$

and condition (iv) holds.

Now that the Fourier conditions have been verified the only thing remaining is to show that the initial estimates provided in equations (18) and (20) lie in the interval $(M, M + K)$. To show that Pope's estimate lies in this interval the following inequality must hold:

$$(26) \quad M < \log(\exp(M) + K\exp(M/2)) < M + K.$$

This is equivalent to

$$(27) \quad 0 < K\exp(M/2) < \exp(M + K) - \exp(M).$$

Clearly the left inequality holds. So consider the right inequality, or, equivalently, since $\exp(K) - 1 > 0$,

$$(28) \quad K/(\exp(K) - 1) < \exp(M/2).$$

But, again from (25), (28) obviously holds, and hence (27) and (26) are true.

To demonstrate that the initial estimate provided in (20) falls in this interval is somewhat more involved, but is, nonetheless, straight-forward. First the inequality

$$(29) \quad A/(B + \sqrt{B^2 - A}) > M$$

must be verified. But this is equivalent to

$$A - 2BM + M^2 > 0, \text{ if } A - BM > 0.$$

Substituting $A = 12\,(HM + 1)$ and $B = 3\,(2H + 1)$ gives upon simplification

$$M^2 - 6M + 12 > 0$$

which is always true since this quadratic in M has a minimum value of three. Hence, (29) holds provided that $A - BM > 0$. But this is the same as

$$H > (M - 4)/2M$$

which is, since $H > 0$, obviously satisfied for $M < 4$.

Finally the inequality

$$(30) \quad A/(B + \sqrt{B^2 - A}) < M + K$$

must be considered. This is equivalent to the inequality

$$A - 2B(M + K) + (M + K)^2 < 0, \quad \text{if } A - B(M + K) > 0,$$

which upon replacement of A and B, reduces to

$$(M + K)^2 - 6(M + K) < 0.$$

This last inequality holds provided that

$$(31) \quad K < 6 - M$$

which, in turn holds for $M < 4$ and $K < 2$. However, the inequality $A - B(M + K) > 0$ must also be validated. Upon substitution for A and B this becomes

$$2MH^2 + (2 - M)H - 1 > 0.$$

The quadratic in H on the left side can be shown to be positive for $H > 0.5$. Thus, in summation, (29), (30) and (31) hold provided that $M < 4$ and $H > 0.5$ $(K < 2)$.

The requirement $M < 4$ is in practice no restriction. The condition $H > 0.5$ was taken into account in the algorithm presented above by switching to Pope's estimate when this condition is not true.

Example

Catch data for the West Greenland Atlantic cod (*Gadus morhua*) for 1956 through 1966 have been analyzed by Schumacher (1971). Doubleday (1976) used these data to illustrate a least squares approach for obtaining fishing mortality rates given catch at age data. From the catch data presented in Doubleday (1976) the 1951 and 1953 cohorts were selected to demonstrate the effectiveness of the above algorithms. First, results of an application of Virtual Population Analysis (VPA) utilizing Algorithm II are presented in Table 2. Following Schumacher, values of $M = 0.2$ and $F_t = 0.8$ (fishing mortality rate in the last time period) were used for each of these cohorts. In each stage of VPA, fishing mortality rates were calculated to an accuracy of six significant digits (5×10^{-6}). This degree of accuracy was imposed so that the population size would be precise. Note that three or less iterations were required at each stage of VPA to yield this accuracy.

Next Algorithm I was used to solve forward in time with initial stock size taken from the results of the VPA. The number of iterations required in each step is incorporated into Table 2.

Discussion

Newton's Method was chosen for use in the above algorithm because of its rapid convergence even though evaluation of the function and its derivative is required at each step. However, an examination of the derivative in each case indicates that careful programming will require only simple operations for its evaluation.

It should be apparent at this point that Pope's estimate can be used instead of the Padé approximation to obtain a starting value for Newton's Method in both algorithms. However, as already mentioned, and as demonstrated in Table 1, equation

TABLE 2. Catch data for the 1953 and 1951 cohorts of the West Greenland Cod were chosen from Doubleday (1976). With F_t = 0.8 in 1966 and 1965, and with M = 0.2 a VPA was carried out using Algorithm II. The number of iterations required at each stage is listed in the column headed by Backward. Then using the stock sizes calculated in 1956, Algorithm I was used to solve forward in time. The number of iterations needed at each step is given in the last column.

					Number of iterations	
Year	Age	Catch (in 1000's)	Fishing mortality	Population size (in 1000's)	Backward	Forward
(1953 cohort)						
1956	3	209	0.000652	353612	1	1
1957	4	19353	0.076566	289324	1	1
1958	5	15136	0.079054	219419	1	1
1959	6	27411	0.200397	165990	2	2
1960	7	20250	0.223314	111222	2	2
1961	8	23126	0.427468	72836	2	2
1962	9	13772	0.490300	38890	2	2
1963	10	6768	0.477891	19501	2	2
1964	11	4138	0.609802	9900	2	2
1965	12	1864	0.620121	4405	2	2
1966	13	981	0.800000	1940		2
(1951 cohort)						
1956	5	4996	0.088252	65161	1	1
1957	6	9362	0.236548	48843	2	2
1958	7	7501	0.302290	31565	2	2
1959	8	3881	0.252614	19102	2	2
1960	9	2743	0.284961	12148	2	2
1961	10	2333	0.418190	7480	2	2
1962	11	1709	0.621763	4031	2	2
1963	12	1156	1.222824	1772	3	3
1964	13	321	1.648995	427	3	3
1965	14	34	0.800000	67		2

(14) provides a more precise estimate for Z in equation (1) than does equation (5) for a wider range of F and M, while equation (20) yields a better approximation for Z in equation (2) than does equation (18) for smaller values of Z. To determine what effect this has upon the above algorithms, test runs were made with F ranging from 0.1 to 2.0 in steps of 0.1 and with M going from 0.1 to 1.0 in the same increments. When equation (5) was substituted for equation (14) in Algorithm I the number of iterations increased from an average of 2.64 to an average of 3.74. With equation (18) replacing equation (20) in Algorithm II an average of 2.99 iterations were required rather than 2.74. Thus, in each case use of the Padé approximation provided a savings in the number of iterations.

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